Certain Diagonal Equations over Finite Fields

by

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Dedication

To the memory of my mother
Virginia Sze
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Certain Diagonal Equations over Finite Fields

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Abstract

Let $F_{q^t}$ be the finite field with $q^t$ elements and let $F_{q^t}^*$ be its multiplicative group. We study the diagonal equation $ax^{q-1} + by^{q-1} = c$ where $a, b, c \in F_{q^t}^*$. This equation can be written as $x^{q^2-1} + \alpha y^{q^2-1} = \beta$, where $\alpha, \beta \in F_{q^t}^*$. Let $N_t(\alpha, \beta)$ denote the number of solutions $(x, y) \in F_{q^t}^* \times F_{q^t}^*$ of $x^{q^2-1} + \alpha y^{q^2-1} = \beta$ and $I(r; a, b)$ be the number of monic irreducible polynomials $f \in F_q[x]$ of degree $r$ with $f(0) = a$ and $f(1) = b$. We show that $N_t(\alpha, \beta)$ can be expressed in terms of $I(r; a, b)$, where $r \mid t$ and $a, b \in F_q^*$ are related to $\alpha$ and $\beta$. A recursive formula for $I(r; a, b)$ will be given and we illustrate this by computing $I(r; a, b)$ for $2 \leq r \leq 4$. We also show that $N_3(\alpha, \beta)$ can be expressed in terms of the number of monic irreducible cubic polynomials over $F_q$ with prescribed trace and norm. Consequently, $N_3(\alpha, \beta)$ can be expressed in terms of the number of rational points on a certain elliptic curve. We give a proof that given any $a, b \in F_q^*$ and integer $r \geq 3$, there always exists a monic irreducible polynomial $f \in F_q[x]$ of degree $r$ such that $f(0) = a$ and $f(1) = b$. We also use the result on $N_2(\alpha, \beta)$ to construct a new family of planar functions.
1 Introduction

Let $\mathbb{F}_q$ be the finite field with $q$ elements and let $t$ be a positive integer. The multiplicative group of $\mathbb{F}_{q^t}$ is denoted by $\mathbb{F}_{q^t}^*$. The purpose of this thesis is to study the number of solutions $(x, y) \in \mathbb{F}_{q^t}^* \times \mathbb{F}_{q^t}^*$ of the equation

$$ax^{q-1} + by^{q-1} = c,$$

(1.1)

where $a, b, c \in \mathbb{F}_{q^t}^*$. Equation (1.1) is equivalent to

$$x^{q-1} + \alpha y^{q-1} = \beta,$$

(1.2)

where $\alpha = \frac{b}{a}$, $\beta = \frac{c}{a} \in \mathbb{F}_{q^t}^*$. Let $N_t(\alpha, \beta)$ denote the number of solutions $(x, y) \in \mathbb{F}_{q^t}^* \times \mathbb{F}_{q^t}^*$ of (1.2). The number $N_t(\alpha, \beta)$ is related to the number of rational points on the projective Fermat curve

$$C : \ x^{q-1} + \alpha y^{q-1} - \beta z^{q-1} = 0$$

(1.3)

over $\mathbb{F}_{q^t}$. The number of rational points on $C$ is given by

$$|C(\mathbb{F}_{q^t})| = N_t(\alpha, \beta) + k(q - 1),$$

(1.4)

where $k$ is the number of elements in the multiset $\{-\alpha, \beta, \beta/\alpha\}$ which are $(q - 1)$st powers in $\mathbb{F}_{q^t}$. Equation (1.4) was stated in [23].

Equation (1.2) is a special diagonal equation. In general, the number of solutions of a diagonal equation can be expressed in terms of Gaussian sums and estimates for
the number of solutions can be obtained thereafter. However, the exact number of solutions of a diagonal equation is not known except in some special cases. Wolfmann [29] determined the number of solutions of

$$a_1x_1^d + \cdots + a_sx_s^d = b$$

over $\mathbb{F}_{p^{2m}}$ where $d$ is a “special” divisor of $p^{2m} - 1$, meaning that $d \mid p^r + 1$ for some $r \mid m$. Assume $q^t = p^{2m}$ (i.e., $t \mid 2m$ and $q = p^{2m}$). Then

$$(p^r + 1, q - 1) = \begin{cases} p^{(r, \frac{2m}{t})} + 1 & \text{if } \nu_2 \left(\frac{2m}{t}\right) > \nu_2(r), \\ 2 & \text{if } \nu_2 \left(\frac{2m}{t}\right) \leq \nu_2(r) \text{ and } p > 2, \\ 1 & \text{if } \nu_2 \left(\frac{2m}{t}\right) \leq \nu_2(r) \text{ and } p = 2, \end{cases}$$

where $\nu_2$ is the 2-adic order; see [5, Lemma 2.6] and [17, Lemma 5.3]. Thus, $q - 1$ is not a special divisor of $p^{2m} - 1$ except when $q = 2, 2^2$ or 3. Hence, in general, equation (1.2) is not covered the result of [29].

The focus of this thesis is the number $N_t(\alpha, \beta)$. Let $I(r; a, b)$ denote the number of monic irreducible polynomials $f \in \mathbb{F}_q[x]$ of degree $r$ such that $f(0) = a$ and $f(1) = b$. We shall see that $N_t(\alpha, \beta)$ can be expressed in terms of $I(r; a, b)$ where $r \mid t$ and $a, b \in \mathbb{F}_q^*$ are related to $\alpha$ and $\beta$. This reduces our problem to finding $I(r; a, b)$. The problem of counting (monic) irreducible polynomials with prescribed values resembles that of counting (monic) irreducible polynomials with prescribed coefficients; the latter is a well studied topic in finite fields, see for example [4, 12, 13, 25, 28, 30], but the former, to our knowledge, has not attracted much attention. Here arises a natural question: Is $I(r; a, b)$ always positive? Namely, given $r > 0$ and $a, b \in \mathbb{F}_q^*$, does there always exist a monic irreducible polynomial $f \in \mathbb{F}_q[x]$ such that $f(0) = a$ and $f(1) = b$? The answer is obviously negative for $r = 1, 2$, and is obviously positive for $r = 3$. We are able to prove that $I(r; a, b) > 0$ for all $r \geq 4$ and $a, b \in \mathbb{F}_q^*$. In Chapter 2 we will present preliminary results on Gaussian sums and Möbius
Inversion; these are the basic tools of our investigation. In Chapter 3 we look at the
diagonal equations in general and give the number of solutions in terms of Gaussian
sums. In Chapter 4, we consider our main problem, and we shall express $N_t(\alpha, \beta)$ in
terms of $I(r; a, b)$. We will give a recursive formula for $I(r; a, b)$ and computations
of $I(r; a, b)$ for small values of $r$ in Chapter 5. In Chapter 6, we will derive another
formula for $N_3(\alpha, \beta)$ using a different perspective. This new formula allows us to
relate $N_3(\alpha, \beta)$ to the number of irreducible cubics over $\mathbb{F}_q$ with prescribed trace and
norm and further allows us to relate $N_3(\alpha, \beta)$ to a certain elliptic curve. In Chapter
7, we give a proof that asserts the positivity of $I(r; a, b)$, for $t \geq 3$. In the last chapter
we will discuss some application to planar functions which are also known as perfect
linear functions. We will use the result on $N_2(\alpha, \beta)$ to construct a new family of
planar functions.
2 Preliminary Results

2.1 Characters

Let $G$ be a finite abelian group written multiplicatively. A character of $G$ is a map $\chi$ from $G$ into the multiplicative group of complex numbers of absolute value 1 such that

$$\chi(g_1g_2) = \chi(g_1)\chi(g_2) \quad \text{for all } g_1, g_2 \in G.$$ 

Equivalently, a character of a finite abelian group $G$ is a homomorphism $\chi : G \to \mathbb{C}^*$.

If $1_G$ is the identity element in $G$, then $\chi(1_G) = 1$. If $g \in G$, then $\chi(g)$ is a $|G|$th root of unity and $\chi(g^{-1}) = (\chi(g))^{-1} = \overline{\chi(g)}$, where the bar denotes complex conjugation.

For any finite abelian group $G$, we have the trivial character $\chi_0$ defined by $\chi_0(g) = 1$ for all $g \in G$. For each character $\chi$ of $G$, there is associated the conjugate character $\overline{\chi}$ defined by $\overline{\chi}(g) = \overline{\chi(g)}$ for all $g \in G$. Given the characters $\chi_1, \ldots, \chi_n$, we define the product $\chi_1 \cdots \chi_n$ by $(\chi_1 \cdots \chi_n)(g) = \chi_1(g) \cdots \chi_n(g)$. The set $G^\wedge$ of characters of $G$ forms an abelian group under multiplication of characters and $|G| = |G^\wedge|$. In fact, $G \cong G^\wedge$ although the isomorphism is not canonical.

Let $\mathbb{F}_q$ be the finite field with $q$ elements. Then $\mathbb{F}_q$ and $\mathbb{F}_q^*$ are finite abelian groups under addition and multiplication, respectively. Consider first the additive group of $\mathbb{F}_q$. Let $q = p^n$, where $p$ is a prime. Let $\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p} : \mathbb{F}_q \to \mathbb{F}_p$ be the absolute trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$. Then the canonical additive character of $\mathbb{F}_q$, denoted by $\chi_1$, is given by

$$\chi_1(c) = e^{2\pi i \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(c)/p} \quad \text{for all } c \in \mathbb{F}_q. \quad (2.1)$$
For each $b \in \mathbb{F}_q$, the function $\chi_b$ defined by

$$\chi_b(c) = \chi_1(bc) \quad \text{for all } c \in \mathbb{F}_q$$

is also an additive character of $\mathbb{F}_q$ and all additive characters of $\mathbb{F}_q$ are found in this manner. Now, let us consider the multiplicative group $\mathbb{F}_q^*$ of $\mathbb{F}_q$. The characters of $\mathbb{F}_q^*$ are called the multiplicative characters of $\mathbb{F}_q$. Let $g$ be a fixed primitive element of $\mathbb{F}_q$. Then for each $j = 0, 1, \ldots, q - 2$, the function $\psi_j$ defined by

$$\psi_j(g^k) = e^{2\pi ijk/(q-1)} \quad \text{for } k = 0, 1, \ldots, q - 2$$

is a multiplicative character of $\mathbb{F}_q$ and all multiplicative characters of $\mathbb{F}_q$ are obtained in this way. Furthermore, the set of all multiplicative characters of $\mathbb{F}_q$ forms a cyclic group of order $q - 1$.

Let $q$ be odd and $\eta$ be the function on $\mathbb{F}_q^*$ defined by

$$\eta(c) = \begin{cases} 
1 & \text{if } c \text{ is a square in } \mathbb{F}_q^*, \\
-1 & \text{otherwise.}
\end{cases}$$

Then $\eta$ is a multiplicative character of $\mathbb{F}_q$ called the quadratic character of $\mathbb{F}_q$. For convenience, we define $\eta(0) = 0$.

We have the following identities involving the additive and multiplicative characters of $\mathbb{F}_q$. If $\chi_a$ and $\chi_b$ are additive characters of $\mathbb{F}_q$ we have

$$\sum_{c \in \mathbb{F}_q} \chi_a(c)\overline{\chi_b(c)} = \begin{cases} 
0 & \text{if } a \neq b, \\
q & \text{if } a = b.
\end{cases}$$

In particular,

$$\sum_{c \in \mathbb{F}_q} \chi_a(c) = 0 \quad \text{for } a \neq 0.$$
Moreover, if $c, d \in \mathbb{F}_q$ then
\[
\sum_{b \in \mathbb{F}_q} \chi_b(c) \overline{\chi_b(d)} = \begin{cases} 
0 & \text{if } c \neq d, \\
q & \text{if } c = d.
\end{cases}
\] (2.2)

For multiplicative characters $\psi$ and $\tau$ of $\mathbb{F}_q$ we have
\[
\sum_{c \in \mathbb{F}_q^*} \psi(c) \overline{\tau(c)} = \begin{cases} 
0 & \text{if } \psi \neq \tau, \\
q - 1 & \text{if } \psi = \tau.
\end{cases}
\]

In particular,
\[
\sum_{c \in \mathbb{F}_q^*} \psi(c) = 0 \quad \text{for } \psi \neq \psi_0. \tag{2.3}
\]

Furthermore, if $c, d \in \mathbb{F}_q^*$ then
\[
\sum_{\psi} \psi(c) \overline{\psi(d)} = \begin{cases} 
0 & \text{if } c \neq d, \\
q - 1 & \text{if } c = d,
\end{cases}
\] (2.4)

where the sum is over all multiplicative characters $\psi$ of $\mathbb{F}_q$.

Characters are used to find expressions for the number of solutions of equations in a finite abelian group $G$. Let $f(x_1, \ldots, x_n) = b$ be an equation in $n$ indeterminates over $G$. Let $N(b)$ be the number of $(x_1, \ldots, x_n) \in G^n$ such that $f(x_1, \ldots, x_n) = b$. Then
\[
N(b) = \frac{1}{|G|} \sum_{x_1 \in G} \cdots \sum_{x_n \in G} \sum_{\chi \in G^\wedge} \chi(f(x_1, \ldots, x_n)) \overline{\chi(b)}. \tag{2.5}
\]

### 2.2 Gaussian Sums

Let $\psi$ be a multiplicative and $\chi$ be an additive character of $\mathbb{F}_q$. The Gaussian sum $G(\psi, \chi)$ is defined by
\[
G(\psi, \chi) = \sum_{c \in \mathbb{F}_q^*} \psi(c) \chi(c).
\]
Let $\chi_0$ and $\psi_0$ be the trivial additive and multiplicative characters of $\mathbb{F}_q$ respectively. The Gaussian sum $G(\psi, \chi)$ satisfies

$$G(\psi, \chi) = \begin{cases} 
q - 1 & \text{if } \psi = \psi_0, \chi = \chi_0, \\
-1 & \text{if } \psi = \psi_0, \chi \neq \chi_0, \\
0 & \text{if } \psi \neq \psi_0, \chi = \chi_0,
\end{cases} \tag{2.6}$$

and

$$|G(\psi, \chi)| = q^{1/2} \quad \text{if } \psi \neq \psi_0, \chi \neq \chi_0. \tag{2.7}$$

The Gaussian sums for the finite field $\mathbb{F}_q$ also have the following properties:

(i) $G(\psi, \chi_{ab}) = \overline{\psi(a)}G(\psi, \chi_b)$ for $a \in \mathbb{F}_q^*$, $b \in \mathbb{F}_q$;

(ii) $G(\psi, \overline{\chi}) = \psi(-1)G(\psi, \chi)$;

(iii) $G(\overline{\psi}, \chi) = \psi(-1)G(\psi, \chi)$;

(iv) $G(\psi, \chi)G(\overline{\psi}, \chi) = \psi(-1)q$ for $\psi \neq \psi_0$ and $\chi \neq \chi_0$;

(v) $G(\psi^p, \chi_b) = G(\psi, \chi_{\sigma(b)})$ for $b \in \mathbb{F}_q$, where $p$ is the characteristic of $\mathbb{F}_q$ and $\sigma(b) = b^p$.

Let $\psi$ be a multiplicative character of $\mathbb{F}_q$. By (2.2) we have, for any $c \in \mathbb{F}_q^*$

$$\psi(c) = \frac{1}{q} \sum_{d \in \mathbb{F}_q^*} \psi(d) \sum_{b \in \mathbb{F}_q} \chi_b(c)\overline{\chi_b(d)}$$

$$= \frac{1}{q} \sum_{b \in \mathbb{F}_q} \chi_b(c) \sum_{d \in \mathbb{F}_q} \psi(d)\overline{\chi_b(d)}$$

$$= \frac{1}{q} \sum_{\chi} G(\psi, \overline{\chi})\chi(c),$$

where the last sum is extended over all additive characters $\chi$ of $\mathbb{F}_q$. Similarly, if $\chi$ is
an additive character of $\mathbb{F}_q$, then by (2.4), we get, for any $c \in \mathbb{F}_q^*$

$$
\chi(c) = \frac{1}{q-1} \sum_{d \in \mathbb{F}_q^*} \chi(d) \sum_{\psi} \psi(c) \overline{\psi}(d) \\
= \frac{1}{q-1} \sum_{\psi} \psi(c) \sum_{d \in \mathbb{F}_q^*} \overline{\psi}(d) \chi(d) \\
= \frac{1}{q-1} \sum_{\psi} G(\overline{\psi}, \chi) \psi(c),
$$

(2.8)

where the sum is extended over all multiplicative characters $\psi$ of $\mathbb{F}_q$.

### 2.3 Möbius Inversion

A partially ordered set $(S, \leq)$ is an ordered pair consisting of a set $S$ and a binary relation $\leq$ on $S$ that is reflexive, transitive and anti-symmetric. An interval of a partially ordered set $(S, \leq)$ is given by $[x, y] = \{ z \in S : x \leq z \leq y \}$. We say that a partially ordered set is locally finite if every interval has a finite number of elements.

Let $(S, \leq)$ be a locally finite partially ordered set. The Möbius function of $(S, \leq)$ is an integer valued function of two variables on $S$ defined by

$$
\mu(x, y) = 0 \quad \text{if } x \not\leq y,
$$

and by

$$
\sum_{z \in [x, y]} \mu(x, z) = \delta(x, y) \quad \text{if } x \leq y,
$$

where $\delta$ is the Kronecker delta function.

**Theorem 2.1** (Möbius Inversion Formula [1]). Let $(S, \leq)$ be a locally finite partially ordered set with Möbius function $\mu$. Let $A$ be an abelian group and $N_\equiv : S \to A$ be a function. Let $l, m \in S$ be fixed and for $x \in S$ define

$$
N_{\geq}(x) = \sum_{y \in [x, m]} N_\equiv(y)
$$
and

\[ N_\leq(x) = \sum_{y \in [l,x]} N_=(y). \]

Then

\[ N_=(x) = \sum_{y \in [x,m]} \mu(x,y)N_\geq(y) \quad \text{for all } x \in S \text{ with } x \leq m \]

and

\[ N_=(x) = \sum_{y \in [l,x]} \mu(y,x)N_\leq(y) \quad \text{for all } x \in S \text{ with } x \geq l. \]

**Example 2.2.** [Classical Möbius function] Let \( \mathbb{Z}^+ \) be the set of all positive integers. Then \((\mathbb{Z}^+, |)\) is a locally finite partially ordered set, where \( x \mid y \) means \( x \) divides \( y \). The Möbius function is given by

\[ \mu(x, y) = \mu \left( \frac{y}{x} \right) = \begin{cases} 
1 & \text{if } \frac{y}{x} = 1, \\
(-1)^k & \text{if } \frac{y}{x} \text{ is a product of } k \text{ distinct primes}, \\
0 & \text{if } \frac{y}{x} \text{ is divisible by a square of a prime}.
\end{cases} \]

**Example 2.3.** [Partitions of a set [1]] Let \( S_n \) be a finite set consisting of \( n \) elements. Let \( \{\pi_1, \pi_2, \ldots\} \) be a partition of \( S_n \) into subsets of \( S_n \). The sets \( \pi_i \) are called blocks of the partition. Let \( \mathcal{P} \) be the set of all partitions of \( S_n \) and let \( \pi, \sigma \in \mathcal{P} \). We write \( \pi \leq \sigma \) to mean that \( \pi \) is a refinement of \( \sigma \). Then \((\mathcal{P}, \leq)\) is a locally finite partially ordered set. Then the Möbius function is given by

\[ \mu(\pi, \sigma) = (-1)^{r(\sigma) - r(\pi)} \prod_{i=1}^{r(\sigma)} (n_i - 1)! \]

where \( r(\pi) \) denotes the number of blocks of \( \pi \) and the \( i \)th block of \( \sigma \) (for some fixed order) is the union of exactly \( n_i \) blocks of \( \pi \).
A diagonal equation over $\mathbb{F}_q$ is an equation of the form

$$a_1x_1^{k_1} + \ldots + a_nx_n^{k_n} = b,$$

(3.1)

where $k_1, \ldots, k_n$ are positive integers, $a_1, \ldots, a_n \in \mathbb{F}_q^*$ and $b \in \mathbb{F}_q$. In this chapter, we will use Gaussian sums to express the number of solutions of diagonal equations.

Let $N$ be the number of solutions of (3.1) in $\mathbb{F}_q^n$. By (2.5) we have

$$N = \frac{1}{q} \sum_{c_1, \ldots, c_n \in \mathbb{F}_q} \chi(a_1c_1^{k_1} + \ldots + a_nc_n^{k_n})\overline{\chi}(b),$$

where $\chi$ runs through all the additive character of $\mathbb{F}_q$. Rearranging and separating the trivial character $\chi_0$, we get

$$N = \frac{1}{q} \sum_{s \in \mathbb{F}_q^*} \overline{\chi}(b) \sum_{c_1, \ldots, c_n \in \mathbb{F}_q} \chi_s(a_1c_1^{k_1}) \cdots \chi_s(a_nc_n^{k_n})$$

$$= \frac{1}{q} (q^n) + \frac{1}{q} \sum_{s \in \mathbb{F}_q^*} \overline{\chi}(b) \sum_{c_1, \ldots, c_n \in \mathbb{F}_q} \chi_s(a_1c_1^{k_1}) \cdots \chi_s(a_nc_n^{k_n})$$

$$= q^{n-1} + \frac{1}{q} \sum_{s \in \mathbb{F}_q^*} \overline{\chi}(b) \left( \sum_{c_1 \in \mathbb{F}_q} \chi_{a_1s}(c_1^{k_1}) \right) \cdots \left( \sum_{c_n \in \mathbb{F}_q} \chi_{a_ns}(c_n^{k_n}) \right).$$

We look at the sum $\sum_{c_i \in \mathbb{F}_q} \chi_{a_is}(c_i^{k_i})$. By (2.8),

$$\chi_{a_1s}(c_1^{k_1}) = \frac{1}{q - 1} \sum_{\psi} G(\overline{\psi}, \chi_{a_1s})\psi(c_1^{k_1}),$$
where the sum is over all multiplicative characters $\psi$ of $\mathbb{F}_q$. We have
\[
\sum_{c_i \in \mathbb{F}_q^*} \chi_{a_s}(c_i^{k_i}) = 1 + \sum_{c_i \in \mathbb{F}_q^*} \chi_{a_s}(c_i^{k_i}) \\
= 1 + \frac{1}{q - 1} \sum_{c_i \in \mathbb{F}_q^*} \sum_{\psi} G(\overline{\psi}, \chi_{a_s}) \psi(c_i^{k_i}) \\
= 1 + \frac{1}{q - 1} \sum_{\psi} G(\overline{\psi}, \chi_{a_s}) \sum_{c_i \in \mathbb{F}_q^*} \psi^{k_i}(c_i).
\]

By (2.3),
\[
\sum_{c_i \in \mathbb{F}_q^*} \psi^{k_i}(c_i) = \begin{cases} 
q - 1 & \text{if } \psi^{k_i} = \psi_0, \\
0 & \text{if } \psi^{k_i} \neq \psi_0,
\end{cases}
\]
where $\psi_0$ is the trivial multiplicative character of $\mathbb{F}_q$. Now let $d_i = \gcd(k_i, q - 1)$. Then $\psi^{k_i}$ is trivial if and only if $o(\psi) \mid d_i$, where $o(\psi)$ is the order of $\psi$. Let $\lambda_i$ be a multiplicative character of order $d_i$. Since $\lambda_i$ is of order $d_i$, then the characters whose order divides $d_i$ are exactly given by $\overline{\lambda_i}^{j_i}$, for $j_i = 0, 1, \ldots, d_i - 1$. Hence,
\[
\sum_{c_i \in \mathbb{F}_q} \chi_{a_s}(c_i^{k_i}) = 1 + \frac{1}{q - 1} \sum_{j_i=0}^{d_i-1} G(\lambda_i^{j_i}, \chi_{a_s}) \sum_{c_i \in \mathbb{F}_q^*} \overline{\lambda_i}^{j_i}(c_i) \\
= 1 + \sum_{j_i=0}^{d_i-1} G(\lambda_i^{j_i}, \chi_{a_s})
\]

Finally, by (2.6) and property (i) of Gaussian sums we get
\[
\sum_{c_i \in \mathbb{F}_q} \chi_{a_s}(c_i^{k_i}) = \sum_{j_i=1}^{d_i-1} G(\lambda_i^{j_i}, \chi_{a_s}) \\
= \sum_{j_i=1}^{d_i-1} \overline{\lambda_i}^{j_i}(a_i) G(\lambda_i^{j_i}, \chi_s)
\]
Therefore,

\[ N = q^{n-1} + \frac{1}{q} \sum_{s \in \mathbb{F}_q^*} \overline{\chi}_s(b) \left( \sum_{j_1=1}^{d_1-1} \lambda_1^{j_1} (a_1) G(\lambda_1^{j_1}, \chi_s) \right) \cdots \left( \sum_{j_n=1}^{d_n-1} \lambda_n^{j_n} (a_n) G(\lambda_n^{j_n}, \chi_s) \right) \]

\[ = q^{n-1} + \frac{1}{q} \sum_{j_1=1}^{d_1-1} \cdots \sum_{j_n=1}^{d_n-1} \lambda_1^{j_1} (a_1) \cdots \lambda_n^{j_n} (a_n) \sum_{s \in \mathbb{F}_q^*} \overline{\chi}_s(b) G(\lambda_1^{j_1}, \chi_s) \cdots G(\lambda_n^{j_n}, \chi_s). \]

For the inner sum, we have

\[ \sum_{s \in \mathbb{F}_q^*} \overline{\chi}_s(b) G(\lambda_1^{j_1}, \chi_s) \cdots G(\lambda_n^{j_n}, \chi_s) = G(\lambda_1^{j_1}, \chi_1) \cdots G(\lambda_n^{j_n}, \chi_1) \sum_{s \in \mathbb{F}_q^*} \overline{\chi}_b(a) \lambda_1^{j_1} (a) \cdots \lambda_n^{j_n} (a) \]

\[ = G(\lambda_1^{j_1}, \chi_1) \cdots G(\lambda_n^{j_n}, \chi_1) G(\lambda_1^{j_1} \cdots \lambda_n^{j_n}, \chi_b). \]

Thus,

\[ N = q^{n-1} + \frac{1}{q} \sum_{j_1=1}^{d_1-1} \cdots \sum_{j_n=1}^{d_n-1} \lambda_1^{j_1} (a_1) G(\lambda_1^{j_1}, \chi_1) \cdots \lambda_n^{j_n} (a_n) G(\lambda_n^{j_n}, \chi_1) G(\lambda_1^{j_1} \cdots \lambda_n^{j_n}, \chi_b) \]

\[ = q^{n-1} + \frac{1}{q} \sum_{j_1=1}^{d_1-1} \cdots \sum_{j_n=1}^{d_n-1} G(\lambda_1^{j_1}, \chi_{a_1}) \cdots G(\lambda_n^{j_n}, \chi_{a_n}) G(\lambda_1^{j_1} \cdots \lambda_n^{j_n}, \chi_b). \]

(3.2)
4 The Main Problem

Let $\mathbb{F}_q$ be the finite field with $q$ elements and let $t$ be a positive integer. Consider the equation

$$x^{q-1} + \alpha y^{q-1} = \beta,$$

where $\alpha, \beta \in \mathbb{F}_q^*$. We want to know the number of solutions $(x, y) \in \mathbb{F}_q^* \times \mathbb{F}_q^*$ of (4.1). Let

$$N_t(\alpha, \beta) = \left| \{(x, y) \in \mathbb{F}_q^* \times \mathbb{F}_q^* : x^{q-1} + \alpha y^{q-1} = \beta \} \right|$$

and for $a, b \in \mathbb{F}_q^*$ and $r \geq 1$, let

$$I(r; a, b) = \left| \{ f \in \mathbb{F}_q[x] : f \text{ monic, \ irred. \ } \deg f = r, f(0) = a, f(1) = b \} \right|.$$

We give a formula for $N_t(\alpha, \beta)$ in terms of $I(r; a, b)$ where $r | t$ and $a, b \in \mathbb{F}_q^*$ are related to $\alpha$ and $\beta$. For any integer $s$, let $\mathbb{F}_q^{*(s)}$ be the group defined by

$$\mathbb{F}_q^{*(s)} = \{ x^s : x \in \mathbb{F}_q^* \}.$$ 

We denote the norm function from $\mathbb{F}_q^t$ to $\mathbb{F}_q$ by $N_{\mathbb{F}_q^t/\mathbb{F}_q}$.

**Theorem 4.1.** For $\alpha, \beta \in \mathbb{F}_q^*$,

$$N_t(\alpha, \beta) = (q - 1)^2 \sum_{r \mid t} \sum_{a, b \in \mathbb{F}_q^*} r \left( \sum_{\alpha, \beta \in \mathbb{F}_q^{*(s-1/r)}} I(r; a, b) \right),$$

where $s = \frac{t}{r}$ and $\frac{a}{a^{1/r}} = N_{\mathbb{F}_q^t/\mathbb{F}_q}(\alpha)$, $\frac{b}{b^{t/r}} = N_{\mathbb{F}_q^t/\mathbb{F}_q}(\beta)$.
Proof. Put \( \mathcal{X} = \{(x, y) \in \mathbb{F}^*_{q^t} \times \mathbb{F}^*_{q^t} : x + \alpha y = \beta \} \). Then we have

\[
N_t(\alpha, \beta) = (q - 1)^2|\mathcal{X}|. \tag{4.2}
\]

Let

\[
\mathcal{U} = \{u \in \mathbb{F}^*_{q^t} : N_{\mathbb{F}_{q^t}/\mathbb{F}_q}(u) = N_{\mathbb{F}_{q^t}/\mathbb{F}_q}(\alpha), \ N_{\mathbb{F}_{q^t}/\mathbb{F}_q}(u + 1) = N_{\mathbb{F}_{q^t}/\mathbb{F}_q}(\beta)\}.
\]

We claim that the mapping

\[
\phi : \mathcal{X} \rightarrow \mathcal{U} \quad (x, y) \mapsto \frac{\alpha y}{x}
\]

is a bijection. Let \( x + \alpha y = \beta \). Then

\[
N_{\mathbb{F}_{q^t}/\mathbb{F}_q}\left(\frac{\alpha y}{x}\right) = \left(\frac{\alpha y}{x}\right)^{(q^t-1)/(q-1)} = \alpha^{(q^t-1)/(q-1)} = N_{\mathbb{F}_{q^t}/\mathbb{F}_q}(\alpha).
\]

Similarly,

\[
N_{\mathbb{F}_{q^t}/\mathbb{F}_q}\left(\frac{\alpha y}{x} + 1\right) = N_{\mathbb{F}_{q^t}/\mathbb{F}_q}\left(\frac{\beta}{x}\right) = N_{\mathbb{F}_{q^t}/\mathbb{F}_q}(\beta).
\]

This shows that \( \phi \) is well-defined. Let \( x_1 + \alpha y_1 = \beta = x_2 + \alpha y_2 \) and \( \frac{\alpha y_1}{x_1} = \frac{\alpha y_2}{x_2} \), then clearly \((x_1, y_1) = (x_2, y_2)\). Thus \( \phi \) is one-to-one. To show that \( \phi \) is onto, let \( u \in \mathcal{U} \).

Then \( \frac{\beta}{1+u}; \frac{\beta u}{\alpha(1+u)} \in \mathbb{F}^*_{q^t-1} \) since

\[
\left(\frac{\beta}{1+u}\right)^{(q^t-1)/(q-1)} = \frac{N_{\mathbb{F}_{q^t}/\mathbb{F}_q}(\beta)}{N_{\mathbb{F}_{q^t}/\mathbb{F}_q}(1+u)} = 1
\]

and

\[
\left(\frac{\beta u}{\alpha(1+u)}\right)^{(q^t-1)/(q-1)} = \frac{N_{\mathbb{F}_{q^t}/\mathbb{F}_q}(\beta)N_{\mathbb{F}_{q^t}/\mathbb{F}_q}(u)}{N_{\mathbb{F}_{q^t}/\mathbb{F}_q}(\alpha)N_{\mathbb{F}_{q^t}/\mathbb{F}_q}(1+u)} = 1.
\]

Furthermore,

\[
\frac{\beta}{1+u} + \alpha \left(\frac{\beta u}{\alpha(1+u)}\right) = \beta.
\]
Therefore we have shown that \( \phi \) is a bijection with inverse

\[
\phi^{-1} : U \rightarrow X
\]

\[
u \mapsto \frac{\beta}{1+u} \left( 1, \frac{u}{\alpha} \right).
\]

Hence,

\[|X| = |U|. \quad (4.3)\]

Let \( U_r = \{ u \in U : [F_q(u) : F_q] = r \} \). Then \( |U| = \sum_r |U_r| \). Let \( u \in F_{q^r} \) such that \([F_q(u) : F_q] = r \) and let \( f \in F_q[x] \) be the minimal polynomial of \( u \) over \( F_q \). We have

\[
N_{F_{q^r}/F_q}(u) = N_{F_{q^r}/F_q}(N_{F_{q^r}/F_{q^r}}(u)) = N_{F_{q^r}/F_q}(u^{t/r}) = [(-1)^r f(0)]^{t/r}.
\]

Similarly, since \( f(x-1) \) is the minimal polynomial of \( u + 1 \) over \( F_q \) we have

\[
N_{F_{q^r}/F_q}(u) = [(-1)^r f(-1)]^{t/r}.
\]

Let

\[
\mathcal{I}_r = \{ f \in F_q[x] : f \text{ monic, irr. deg } f = r, f(0)^{t/r} = N_{F_{q^r}/F_q}(\alpha), f(1)^{t/r} = N_{F_{q^r}/F_q}(\beta) \}.
\]

Then it is clear that the mapping

\[
U_r \rightarrow \mathcal{I}_r
\]

\[
u \mapsto (-1)^r f(-x),
\]

where \( f \) is the minimal polynomial of \( u \) over \( F_q \), is onto and \( r \)-to-1. So \( |U_r| = r|\mathcal{I}_r| \).

From (4.4) we see that \( \mathcal{I}_r = \emptyset \) unless \( N_{F_{q^r}/F_q}(\alpha), N_{F_{q^r}/F_q}(\beta) \in F_{q^{s(t/r)}} \).

We first claim that \( F_{q^{s(t/r)}} = F_{q^{s(q-1,t/r)}} \), where \( (q-1, t/r) = \gcd(q-1, t/r) \). Clearly, \( F_{q^{s(t/r)}} \subset F_{q^{s(q-1,t/r)}} \). If \( \alpha \in F_{q^{s(q-1,t/r)}} \), then \( \alpha = x^{a(q-1)+b(t/r)} \) for some \( x \in F_q^* \) and
integers $a, b$. And so $\alpha = x^{b(t/r)} \in \mathbb{F}_q^{*}(t/r)$. Next, we show that $N_{\mathbb{F}_q^{t/r} / \mathbb{F}_q}(\alpha) \in \mathbb{F}_q^{*}(t/r)$ if and only if $\alpha \in \mathbb{F}_q^{*}(q^{-1}, t/r)$.

\[
N_{\mathbb{F}_q^{t/r} / \mathbb{F}_q}(\alpha) \in \mathbb{F}_q^{*}(t/r) \iff \alpha^{q^{-1} - 1} \in \mathbb{F}_q^{*}(t/r) = \mathbb{F}_q^{*}(q^{-1}, t/r) \iff \left( \frac{\alpha^{q^{-1} - 1}}{(t/r, q^{-1})} \right) = 1 \iff \alpha^{(t/r, q^{-1})} = 1 \iff \alpha \in \mathbb{F}_q^{*}(q^{-1}, t/r).
\]

Therefore, $\mathcal{I}_r = \emptyset$ unless $\alpha, \beta \in \mathbb{F}_q^{*}(q^{-1}, t/r)$. And so we get

\[
|\mathcal{U}| = \sum_{r \mid t} |\mathcal{U}_r| = \sum_{r \mid t} r |\mathcal{I}_r| = \sum_{r \mid t} r \sum_{\alpha, \beta \in \mathbb{F}_q^{*}(q^{-1}, t/r)} I(r; a, b).
\]

(4.5)

The conclusion follows from (4.2), (4.3), and (4.5).

\[\blacksquare\]
5 \textbf{Number of Irreducible Polynomials with Prescribed Values}

In this chapter, we give a recursive formula for $I(r; a, b)$. We also give explicit formulas for $I(r; a, b)$ for $r = 2, 3, 4$.

\section*{5.1 A Recursive Formula for $I(r; a, b)$}

For integer $r > 0$ and $a, b \in \mathbb{F}_q^*$, let

$$ I(r; a, b) = \{f \in \mathbb{F}_q[x] : f \text{ monic, irr. deg } f = r, f(0) = a, f(1) = b\}. $$

So $I(r; a, b) = |I(r; a, b)|$. If $f \in I(1; a, b)$, then $f = (b - a)x + a$. So

$$ I(1; a, b) = \begin{cases} 1 & \text{if } b - a = 1, \\ 0 & \text{otherwise}. \end{cases} \quad (5.1) $$

For integer $i > 0, \lambda \geq 0$ and $a = (a, b) \in \mathbb{F}_q^* \times \mathbb{F}_q^*$, let

$$ I^\lambda(i; a) = \{f_1 \cdots f_\lambda : f_1, \ldots, f_\lambda \in \mathbb{F}_q[x] \text{ monic, irr. of deg } i, \\ (f_1 \cdots f_\lambda)(0) = a, (f_1 \cdots f_\lambda)(1) = b\} $$
and let $I^\lambda(i; a) = |\mathcal{I}^\lambda(i; a)|$. We define $I^0(i; a) = 1$, and write $I^1(i; a) = I(i; a)$. We have

$$q^{r-2} = \left| \{ f \in \mathbb{F}_q[x] : f \text{ monic of deg } r, f(0) = a, f(1) = b \} \right| = \sum_{1 \leq s \leq \lambda} n_s(\tau) \prod_{i=1}^r \prod_{a_i \in \mathbb{F}_q^*} I^\lambda_i(i; a_i).$$

And so

$$I(r; a, b) = q^{r-2} - \sum_{1 \leq s \leq \lambda} n_s(\tau) \prod_{a \in \mathbb{F}_q^*} I^\lambda(\tau; a). \quad (5.2)$$

In the following Lemma, we will express $I^\lambda(i; a)$ in terms of $I(i; a')$ where $a' \in \mathbb{F}_q^* \times \mathbb{F}_q^*$. A partition of an integer $\lambda \geq 0$ is a sequence of integers $\tau = (\tau_1, \ldots, \tau_k)$ such that $\tau_1 \geq \cdots \geq \tau_k \geq 1$ and $\tau_1 + \cdots + \tau_k = \lambda$. We write $\tau \vdash \lambda$ to mean that $\tau$ is a partition of $\lambda$. For $\tau = (\tau_1, \ldots, \tau_k) \vdash \lambda$, let

$$n_s(\tau) = |\{ j : \tau_j = s \}|, \quad 1 \leq s \leq \lambda.$$

**Lemma 5.1.** For $i > 0$, $\lambda \geq 0$ and $a \in \mathbb{F}_q^* \times \mathbb{F}_q^*$, we have

$$I^\lambda(i; a) = \sum_{\tau=(\tau_1,\ldots,\tau_k)\vdash \lambda} \frac{1}{n_1(\tau)! \cdots n_\lambda(\tau)!} \sum_{a_1 \in \mathbb{F}_q^* \times \mathbb{F}_q^* \text{ distinct}} \prod_{j=1}^k \left( I(i; a_j) + \tau_j - 1 \right). \quad (5.3)$$

**Proof.** Since the elements of $\mathcal{I}^\lambda(i; a)$ are products of $\lambda$ irreducible polynomials, we partition $\mathcal{I}^\lambda(i; a)$ by looking at the images of 0 and 1 under each irreducible factor and group the elements of $\mathcal{I}^\lambda(i; a)$ having the same set of images, counting multiplicities. 

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So for each $\tau = (\tau_1, \ldots, \tau_k) \vdash \lambda$, let

$$I^\lambda(i; a) = \{ f_1 \cdots f_\lambda : \exists a_1, \ldots, a_\lambda \in F_q^* \times F_q^* \text{ distinct such that } a_1^{\tau_1} \cdots a_\lambda^{\tau_k} = a$$

and $f_s \in I(i; a_j)$ for $\tau_1 + \cdots + \tau_{j-1} < s \leq \tau_1 + \cdots + \tau_j\}.$

Then we have

$$I^\lambda(i; a) = \bigcup_{\tau \vdash \lambda} I^\lambda_\tau(i; a). \quad (5.4)$$

Now fix $\tau = (\tau_1, \ldots, \tau_k) \vdash \lambda$. For $a_1, \ldots, a_\lambda \in F_q^* \times F_q^*$, let

$$J(a_1, \ldots, a_\lambda) = \{ (g_1, \ldots, g_k) : g_j \text{ is a product of } \tau_j \text{ (not necessarily distinct)}$$

$$\text{ elements of } I(i; a_j) \}. $$

Then

$$|J(a_1, \ldots, a_\lambda)| = \prod_{j=1}^k \left( I(i; a_j) + \tau_j - 1 \right). \quad (5.5)$$

Moreover, the mapping

$$\psi : \bigcup_{\begin{subarray}{c} a_1, \ldots, a_\lambda \in F_q^* \times F_q^* \text{ distinct} \\ a_1^{\tau_1} \cdots a_\lambda^{\tau_k} = a \end{subarray}} J(a_1, \ldots, a_\lambda) \longrightarrow I^\lambda(i; a)$$

$$(g_1, \ldots, g_k) \longmapsto g_1 \cdots g_k$$

is $n_1(\tau)! \cdots n_\lambda(\tau)!$-to-1 and onto. Hence

$$|I^\lambda_\tau(i; a)| = \frac{1}{n_1(\tau)! \cdots n_\lambda(\tau)!} \sum_{\begin{subarray}{c} a_1, \ldots, a_\lambda \in F_q^* \times F_q^* \text{ distinct} \\ a_1^{\tau_1} \cdots a_\lambda^{\tau_k} = a \end{subarray}} |J(a_1, \ldots, a_\lambda)|$$

$$= \frac{1}{n_1(\tau)! \cdots n_\lambda(\tau)!} \sum_{\begin{subarray}{c} a_1, \ldots, a_\lambda \in F_q^* \times F_q^* \text{ distinct} \\ a_1^{\tau_1} \cdots a_\lambda^{\tau_k} = a \end{subarray}} \prod_{j=1}^k \left( I(i; a_j) + \tau_j - 1 \right). \quad (5.6)$$

The conclusion follows from (5.4) and (5.6).
With this Lemma, (5.2) becomes a recursive formula for \( I(r; a, b) \). However, in the inner sum of (5.3), the requirement that \( a_1, \ldots, a_k \) be distinct is difficult to implement in actual computation. We shall use a Möbius inversion to waive this requirement.

Fix \( \tau = (\tau_1, \ldots, \tau_k) \vdash \lambda \) and let

\[
A = \{(a_1, \ldots, a_k) : a_1, \ldots, a_k \in \mathbb{F}_q^* \times \mathbb{F}_q^*, a_1^{\tau_1} \cdots a_k^{\tau_k} = a\}.
\]

For each \( (a_1, \ldots, a_k) \in A \) we define an equivalence relation on the set \( \{1, \ldots, k\} \) as follows: \( i \sim j \) if and only if \( a_i = a_j \). We denote the induced partition on \( \{1, \ldots, k\} \) by \( \pi(a_1, \ldots, a_k) \). Let \( P_k \) be the set of all partitions of \( \{1, \ldots, k\} \). For \( \pi, \sigma \in P_k \), we write \( \pi \leq \sigma \) to mean that \( \pi \) is a refinement of \( \sigma \). Then \( (P_k, \leq) \) is a partially ordered set whose smallest element is \( \pi_0 = \{\{1\}, \ldots, \{k\}\} \). For \( \pi \in P_k \), put

\[
\mathcal{A}_\pi = \{(a_1, \ldots, a_k) \in A : \pi(a_1, \ldots, a_k) = \pi\}.
\]

Let \( \pi_1, \ldots, \pi_l \) be the blocks of \( \pi \in P_k \). We have

\[
\sum_{\sigma \geq \pi} \sum_{(a_1, \ldots, a_k) \in \mathcal{A}_\pi} |\mathcal{J}(a_1, \ldots, a_k)| = \sum_{(a_1, \ldots, a_k) \in A} |\mathcal{J}(a_1, \ldots, a_k)|
\]

\[
= \prod_{s=1}^l \prod_{j \in \pi_s} \left( I(i; a_s) + \tau_j - 1 \right).
\]

By the Möbius inversion formula,

\[
\sum_{(a_1, \ldots, a_k) \in A_{\pi_0}} |\mathcal{J}(a_1, \ldots, a_k)| = \sum_{\pi = \{\pi_1, \ldots, \pi_l\} \in P_k} \mu(\pi) \prod_{s=1}^l \prod_{j \in \pi_s} \left( I(i; a_s) + \tau_j - 1 \right).
\]
where $\mu(\pi) = \mu(\pi_0, \pi)$ and $\mu$ is the Möbius function of $(\mathcal{P}_k, \leq)$. By (2.9),

$$
\mu(\pi) = \mu(\pi_0, \pi) = (-1)^{k-l} \prod_{s=1}^{l} (|\pi_s| - 1)!. 
$$

But

$$
A_{\pi_0} = \{(a_1, \ldots, a_k) : a_1, \ldots, a_k \in \mathbb{F}_q^* \times \mathbb{F}_q^* \text{ distinct, } a_1^{\tau_1} \cdots a_k^{\tau_k} = a\}. 
$$

And so by (5.5),

$$
\sum_{(a_1, \ldots, a_k) \in A_{\pi_0}} |\mathcal{J}(a_1, \ldots, a_k)| = \sum_{a_1, \ldots, a_k \in \mathbb{F}_q^* \times \mathbb{F}_q^* \text{ distinct, } a_1^{\tau_1} \cdots a_k^{\tau_k} = a} \prod_{j=1}^{k} \left( I(i; a_j) + \tau_j - 1 \right). 
$$

Hence, we can write (5.3) as

$$
I^\lambda(i; a) = \sum_{\tau=(\tau_1, \ldots, \tau_k) \in \lambda} \frac{1}{n_1(\tau)! \cdots n_\lambda(\tau)!} \sum_{\pi=(\pi_1, \ldots, \pi_l) \in \mathcal{P}_k} \mu(\pi) \prod_{s=1}^{l} \prod_{j \in \pi_s} \left( I(i; a_s) + \tau_j - 1 \right). \tag{5.7}
$$

### 5.2 The Case $r = 2$

In this section we will find an expression for $I(2; a, b)$ and consequently, for $N_2(\alpha, \beta)$. Let $f \in \mathbb{F}_q[x]$ be monic with $\deg f = 2, f(0) = a, f(1) = b$. Then $f$ is of the form

$$
f = x^2 + (b - a - 1)x + a.
$$

We find conditions such that $f$ is irreducible.

We first assume that $q$ is odd. Then $f$ is irreducible if and only if $(b - a - 1)^2 - 4a$ is a nonsquare in $\mathbb{F}_q$. Let $\eta$ be the quadratic character of $\mathbb{F}_q$. Note that we define
\( \eta(0) = 0. \) Then

\[
\eta((b - a - 1)^2 - 4a) = \begin{cases} 
1 & \text{if } (b - a - 1)^2 - 4a \text{ is a square in } \mathbb{F}_q^*, \\
-1 & \text{if } (b - a - 1)^2 - 4a \text{ is a nonsquare in } \mathbb{F}_q^*.
\end{cases}
\]

Thus,

\[
I(2; a, b) = \begin{cases} 
\frac{1}{2} [1 - \eta((b - a - 1)^2 - 4a)] & \text{if } (b - a - 1)^2 - 4a \neq 0, \\
0 & \text{if } (b - a - 1)^2 - 4a = 0.
\end{cases} \tag{5.8}
\]

If \( q \) is even, then \( f = x^2 + (b - a - 1)x + a \in \mathbb{F}_q[x] \) is reducible if and only if \( b - a - 1 = 0 \) or there exists \( \gamma \in \mathbb{F}_q \) such that \( \gamma^2 + (b - a - 1)\gamma + a = 0. \) When \( b - a - 1 \neq 0, \)

\[
\gamma^2 + (b - a - 1)\gamma + a = 0 \iff \left( \frac{\gamma}{b-a-1} \right)^2 + \left( \frac{\gamma}{b-a-1} \right) = \frac{a}{(b-a-1)^2} \\
\iff \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2} \left( \frac{a}{(b-a-1)^2} \right) = 0,
\]

by [22, Theorem 2.25]. Therefore \( f \) is irreducible if and only if \( b - a - 1 \neq 0 \) and \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2} \left( \frac{a}{(b-a-1)^2} \right) = 1. \) Now let \( \chi_1 \) be the canonical additive character of \( \mathbb{F}_q. \) Then by (2.1),

\[
\chi_1 \left( \frac{a}{(b-a-1)^2} \right) = (-1)^{\text{Tr}_{\mathbb{F}_q/\mathbb{F}_2} \left( \frac{a}{(b-a-1)^2} \right)}
= \begin{cases} 
1 & \text{if } \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2} \left( \frac{a}{(b-a-1)^2} \right) = 0, \\
-1 & \text{if } \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2} \left( \frac{a}{(b-a-1)^2} \right) = 1.
\end{cases}
\]

Hence,

\[
I(2; a, b) = \begin{cases} 
\frac{1}{2} \left[ 1 - \chi_1 \left( \frac{a}{(b-a-1)^2} \right) \right] & \text{if } b - a - 1 \neq 0, \\
0 & \text{if } b - a - 1 = 0.
\end{cases} \tag{5.9}
\]

We shall use Theorem 4.1 together with (5.8) and (5.9) to determine \( N_2(\alpha, \beta). \)
Let \(a = N_{\mathbb{F}_q^2/\mathbb{F}_q}(\alpha)\) and \(b = N_{\mathbb{F}_q^2/\mathbb{F}_q}(\beta)\). By Theorem 4.1,

\[
\frac{N_2(\alpha, \beta)}{(q-1)^2} = 2I(2; a, b) + \sum_{\substack{a_1, b_1 \in \mathbb{F}_q^* \\ a_1^2 = a, b_1^2 = b}} I(1; a_1, b_1).
\]

Using (5.1),

\[
\sum_{\substack{a_1, b_1 \in \mathbb{F}_q^* \\ a_1^2 = a, b_1^2 = b}} I(1; a_1, b_1) = \left| \{ a_1 \in \mathbb{F}_q^* : a_1^2 = a, (a_1 + 1)^2 = b \} \right|
= \begin{cases} 
1 & \text{if } (b - a - 1)^2 - 4a = 0, \\
0 & \text{otherwise.}
\end{cases} 
\tag{5.10}
\]

Combining (5.10) with (5.8) and (5.9), we get, if \(q\) is odd,

\[
\frac{N_2(\alpha, \beta)}{(q-1)^2} = 1 - \eta((b - a - 1)^2 - 4a). 
\tag{5.11}
\]

If \(q\) is even,

\[
\frac{N_2(\alpha, \beta)}{(q-1)^2} = \begin{cases} 
1 - \chi^1 \left( \frac{a}{(b - a - 1)^2} \right) & \text{if } b - a - 1 \neq 0, \\
1 & \text{if } b - a - 1 = 0.
\end{cases}
\]

5.3 The Case \(r = 3\)

Let \(f \in \mathbb{F}_q[x]\) be monic with \(\deg f = 3, f(0) = a, f(1) = b\). Then

\[
f = x^3 + cx^2 + (b - a - c - 1)x + a,
\]
for some \( c \in \mathbb{F}_q \). Now \( f \) is irreducible if and only if \( f(x) \neq 0 \) for all \( x \in \mathbb{F}_q \setminus \{0,1\} \). Let

\[
V(a,b) = \left\{ \frac{-1}{x^2 - x} (x^3 + (b - a - 1)x + a) : x \in \mathbb{F}_q \setminus \{0,1\} \right\}
\]

(5.12)

Then \( f \) is irreducible if and only if \( c \notin V(a,b) \). Therefore

\[
I(3; a,b) = q - |V(a,b)|.
\]

(5.13)

To determine \( N_3(\alpha, \beta) \), let \( a = N_{\mathbb{F}_q^3/\mathbb{F}_q}(\alpha) \) and \( b = N_{\mathbb{F}_q^3/\mathbb{F}_q}(\beta) \). By Theorem 4.1 and (5.13),

\[
\frac{N_3(\alpha, \beta)}{(q - 1)^2} = 3I(3; a,b) + \sum_{a_1, b_1 \in \mathbb{F}_q^*} I(1; a_1, b_1)_{a_1^3 = a, b_1^3 = b}
\]

(5.14)

\[
= 3(q - |V(a,b)|) + \left| \left\{ a_1 \in \mathbb{F}_q^* : a_1^3 = a, (a_1 + 1)^3 = b \right\} \right|.
\]

We determine \( \left| \left\{ a_1 \in \mathbb{F}_q^* : a_1^3 = a, (a_1 + 1)^3 = b \right\} \right| \) in (5.14) in the next lemma.

**Lemma 5.2.** Let \( a, b \in \mathbb{F}_q^* \).

(i) When \( p \neq 3 \),

\[
\left| \left\{ a_1 \in \mathbb{F}_q^* : a_1^3 = a, (a_1 + 1)^3 = b \right\} \right|
\]

\[
= \begin{cases} 
2 & \text{if } a = 1, b = -1 \text{ and } 3 \mid q - 1, \\
1 & \text{if } b - a + 2 \neq 0 \text{ and } 3(2a + b - 1)(a + 2b + 1) = (b - a + 2)^2(b - a - 1), \\
0 & \text{otherwise.}
\end{cases}
\]
(ii) When $p = 3$, 

$$\left| \{a_1 \in \mathbb{F}_q^* : a_1^3 = a, (a_1 + 1)^3 = b \} \right| = \begin{cases} 1 & \text{if } b = a + 1, \\ 0 & \text{if } b \neq a + 1. \end{cases}$$

Proof. (i) Let $a, b \in \mathbb{F}_q^*$ and $A = \left| \{a_1 \in \mathbb{F}_q^* : a_1^3 = a, (a_1 + 1)^3 = b \} \right|$. If $a_1 \in A$, then $3a_1^2 + 3a_1 + 1 = b - a$. So $a_1^2 + a_1 + 1 = \frac{1}{3}(b - a + 2)$. 

If $b - a + 2 = 0$, then $a_1^3 = 1$. So $a = 1$ and $b = -1$. It follows that $a_1$ is a primitive cube root of unity. But $\mathbb{F}_q^*$ has a primitive cube root of unity if and only if $3 \mid q - 1$. And so if $a = 1, b = -1$ and $3 \mid q - 1$ then $|A| = 2$.

If $b - a + 2 \neq 0$, we have 

$$a_1 - 1 = \frac{a_1^3 - 1}{a_1^2 + a_1 + 1} = \frac{3(a - 1)}{b - a + 2}.$$ 

So 

$$a_1 = \frac{2a + b - 1}{b - a + 2}. $$

If $a_1 = \frac{2a + b - 1}{b - a + 2}$, the equation $a_1^2 + a_1 + 1 = \frac{1}{3}(b - a + 2)$ becomes equivalent to 

$$3(2a + b - 1)(a + 2b + 1) = (b - a + 2)^2(b - a - 1). \quad (5.15)$$

Thus, if (5.15) is satisfied then $|A| = 1$.

(ii) Obvious.
5.4 The Case \( r = 4 \)

In this section we will present an explicit formula for \( I(4; a, b) \). Using (5.2) we have

\[
I(4; a, b) = q^2 - I^3(1; a, b) - \sum_{a_1, a_2 \in \mathbb{F}_q^* \times \mathbb{F}_q^*} I^2(1; a_1) I(2; a_2)
\]

\[
- I^2(2; a, b) - \sum_{a_1, a_2 \in \mathbb{F}_q^* \times \mathbb{F}_q^*} I(1; a_1) I(3; a_2).
\]

In (5.16), we shall use (5.7) to compute \( I^2(1; a) \), \( I^2(2; a, b) \) and \( I^4(1; a, b) \). Note that for \( i = 1, 2 \), then \( I^{(i)}(a) = 0 \) or \( 1 \) and so the binomial coefficient in (5.7) is simplified to

\[
\binom{I^{(i)}(a) + \tau_j - 1}{I^{(i)}(a)} = I^{(i)}(a)
\]

Performing the computations, we get

\[
I^4(1; a, b) = \frac{1}{24} \sum_{a_1, \ldots, a_4 = (a, b)} I(1; a_1) I(1; a_2) I(1; a_3) I(1; a_4)
\]

\[
+ \frac{1}{4} \sum_{a_1^2 a_2 a_3 = (a, b)} I(1; a_1) I(1; a_2) I(1; a_3) + \frac{1}{8} \sum_{a_1^2 a_2^2 = (a, b)} I(1; a_1) I(1; a_2)
\]

\[
+ \frac{1}{3} \sum_{a_1^2 a_2 = (a, b)} I(1; a_1) I(1; a_2) + \frac{1}{4} \sum_{a_1^2 = (a, b)} I(1; a_1).
\]

Each sum in (5.17) represents the number of solutions of a rational equation or some polynomial equations. And so (5.17) can be written as

\[
I^4(1; a, b) = \frac{1}{24} \left| \{(a_1, a_2, a_3, a_4) : a_i \in \mathbb{F}_q^* : (1 + a_1)(1 + a_1)(1 + a_3) \left( 1 + \frac{a}{a_1 a_2 a_3} \right) = b \} \right|
\]

\[
+ \frac{1}{4} \left| \{(a_1, a_2) \in \mathbb{F}_q^* \times \mathbb{F}_q^* : (1 + a_1)^2 (1 + a_2) \left( 1 + \frac{a}{a_1 a_2} \right) = b \} \right|
\]

\[
+ \frac{1}{8} \left| \{(a_1, a_2) \in \mathbb{F}_q^* \times \mathbb{F}_q^* : a_1^2 a_2^2 = a, (1 + a_1)^2 (1 + a_2)^2 = b \} \right|
\]

\[
+ \frac{1}{3} \left| \{a_1 \in \mathbb{F}_q^* : (1 + a_1)^3 \left( 1 + \frac{a}{a_1^2} \right) = b \} \right| + \frac{1}{4} \left| \{a_1 \in \mathbb{F}_q^* : a_1^4 = a, (1 + a_1)^4 = b \} \right|
\]

(5.18)
Next, for $i = 1, 2$, $a \in \mathbb{F}_q^* \times \mathbb{F}_q^*$ we have

$$I^2(i; a) = \frac{1}{2} \sum_{a_1a_2=a} I(i; a_1)I(i; a_2) + \frac{1}{2} \sum_{a_1^2=a} I(i; a_1). \quad (5.19)$$

Let $i = 1$ and $a = (a, b)$ in (5.19). Then by (5.10),

$$\sum_{a_1^2=a} I(1; a_1) = \begin{cases} 1 & \text{if } (b - a - 1)^2 - 4a = 0, \\ 0 & \text{otherwise}. \end{cases}$$

Now

$$\sum_{a_1a_2=a} I(1; a_1)I(1; a_2) = \left| \left\{ a_1 \in \mathbb{F}_q^* : (1 + a_1)(1 + \frac{a}{a_1}) = b \right\} \right|$$

$$= \left| \left\{ a_1 \in \mathbb{F}_q^* : a_1^2 - (b - a - 1)a_1 + a = 0 \right\} \right|$$

$$= \begin{cases} 1 + \eta((b - a - 1)^2 - 4a) & \text{if } q \text{ is odd}, \\ 1 + \chi \left( \frac{a}{(b - a - 1)^2} \right) & \text{if } q \text{ is even and } b - a - 1 \neq 0, \\ 1 & \text{if } q \text{ is even and } b - a - 1 = 0. \end{cases}$$

Hence, if $q$ is odd, then

$$I^2(1; a) = \begin{cases} 1 & \text{if } \eta((b - a - 1)^2 - 4a) = 0 \text{ or } 1, \\ 0 & \text{if } \eta((b - a - 1)^2 - 4a) = -1, \end{cases}$$

and so

$$\sum_{a_1, a_2 \in \mathbb{F}_q^* \times \mathbb{F}_q^* \atop a_1a_2=(a,b)} I^2(1; a_1)I(2; a_2) = \left| \left\{ (a_1, b_1) \in \mathbb{F}_q^* \times \mathbb{F}_q^* : \eta ((b_1 - a_1 - 1)^2 - 4a_1) = 0 \text{ or } 1; \right. \\ \eta \left( \frac{b}{b_1} - \frac{a}{a_1} - 1 \right)^2 - 4 \frac{a}{a_1} = -1 \right\} \right|. \quad (5.20)$$

Hence, if $q$ is odd, then

$$I^2(1; a) = \begin{cases} 1 & \text{if } \eta((b - a - 1)^2 - 4a) = 0 \text{ or } 1, \\ 0 & \text{if } \eta((b - a - 1)^2 - 4a) = -1, \end{cases}$$

and so

$$\sum_{a_1, a_2 \in \mathbb{F}_q^* \times \mathbb{F}_q^* \atop a_1a_2=(a,b)} I^2(1; a_1)I(2; a_2) = \left| \left\{ (a_1, b_1) \in \mathbb{F}_q^* \times \mathbb{F}_q^* : \eta ((b_1 - a_1 - 1)^2 - 4a_1) = 0 \text{ or } 1; \right. \\ \eta \left( \frac{b}{b_1} - \frac{a}{a_1} - 1 \right)^2 - 4 \frac{a}{a_1} = -1 \right\} \right|. \quad (5.20)$$

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Now if \( q \) is even, then

\[
I^2(1; a) = \begin{cases} 
0 & \text{if } b - a - 1 \neq 0 \text{ and } \chi_1\left(\frac{a}{(b - a - 1)^2}\right) = -1, \\
1 & \text{otherwise.}
\end{cases}
\]

and so we have

\[
\sum_{a_1, a_2 \in F_q^* \times F_q^*} I^2(1; a_1) I(2; a_2) = \left| \left\{ (a_1, b_1) \in F_q^* \times F_q^* : b_1 - a_1 - 1 = 0 \text{ or } \chi_1\left(\frac{a_1}{(b_1 - a_1 - 1)^2}\right) = 1; \right. \right.
\]

\[
\chi_1\left(\frac{b}{b_1 - a_1 - 1}\right) = -1 \right\} \right|.
\]

(5.21)

Now let \( i = 2 \) and \( a = (a, b) \) in (5.19). Then

\[
I^2(2; a, b) = \frac{1}{2} \sum_{a_1, a_2 = (a, b)} I(2; a_1) I(2; a_2) + \frac{1}{2} \sum_{a_1^2 = (a, b)} I(2; a_1).
\]

When \( q \) is odd, we have

\[
I^2(2; a, b) =
\]

\[
\frac{1}{2} \left| \left\{ (a_1, b_1) \in F_q^* \times F_q^* : \eta\left(\left(\frac{b_1 - a_1 - 1}{a_1}\right)^2 - 4a_1\right) = -1, \eta\left(\frac{b_1}{a_1} - \frac{a}{a_1 - 1}\right)^2 - 4\frac{a}{a_1} = -1 \right\} \right| + \frac{1}{2} \left| \left\{ (a_1, b_1) \in F_q^* \times F_q^* : a_1^2 = a, b_1^2 = b, \eta\left(\left(\frac{b_1 - a_1 - 1}{a_1}\right)^2 - 4a_1\right) = -1 \right\} \right|.
\]

(5.22)
When $q$ is even, we have

$$I^2(2; a, b) =$$

$$\frac{1}{2} \left| \left\{ (a_1, b_1) \in \mathbb{F}_q^* \times \mathbb{F}_q^* : \chi_1 \left( \frac{a_1}{(b_1 - a_1 - 1)^2} \right) = -1, \chi_1 \left( \frac{\frac{a}{a_1}}{(\frac{b}{b_1} - \frac{a}{a_1} - 1)^2} \right) = -1 \right\} \right|$$

$$+ \begin{cases} \frac{1}{2} & \text{if } \chi_1 \left( \frac{a_1}{(b_1 - a_1 - 1)^2} \right) = -1, \\ 0 & \text{otherwise.} \end{cases}$$

(5.23)

The last sum in (5.16) is given by

$$\sum_{\substack{a_1, a_2 \in \mathbb{F}_q^* \times \mathbb{F}_q^* \\ a_1 \cdot a_2 = (a, b)}} I(1; a_1)I(3; a_2) = \sum_{a_1 \in \mathbb{F}_q^*, a_1 \neq -1} \left( q - \left| V \left( \frac{a}{a_1}, \frac{b}{a_1 + 1} \right) \right| \right),$$

(5.24)

where $V \left( \frac{a}{a_1}, \frac{b}{a_1 + 1} \right)$ is defined in (5.12).

Now, Equation (5.16) combined with (5.18) and (5.20)–(5.24), is the most explicit formula for $I(4; a, b)$ that this method can offer.
In a recent paper [23], Moisio found a formula for $N_3(\alpha, \beta)$ in terms of the number of rational points on a projective cubic curve. Let $a = N_{F_{q^3}/F_q}(\alpha)$ and $b = N_{F_{q^3}/F_q}(\beta)$. Let $\mathcal{A}$ be the affine cubic curve defined by

$$\mathcal{A} : \ ax^2y + axy^2 + x^2 + ay^2 + (a + 1 - b)xy + x + y = 0$$

and let $\bar{\mathcal{A}}$ be the projective closure of $\mathcal{A}$. Moisio [23, Theorem 2] proved that

$$\frac{N_3(\alpha, \beta)}{(q - 1)^2} = |\bar{\mathcal{A}}(F_q)|,$$

where $\bar{\mathcal{A}}(F_q)$ denotes the set of rational points on $\bar{\mathcal{A}}$ over $F_q$. In this chapter, we will derive another formula for $N_3(\alpha, \beta)$ in terms of the number of rational points on a different (and simpler) projective cubic.

We first define a few terms. For $a, b \in F_q^*$, $a' \in F_q$ and integer $r \geq 1$, let

$$S_r(a, b) = \{u \in F_q^* : N_{F_{q^r}/F_q}(u) = a, \ N_{F_{q^r}/F_q}(u + 1) = b\},$$

$$T_r(a', b) = \{u \in F_q^* : \text{Tr}_{F_{q^r}/F_q}(u) = a', \ N_{F_{q^r}/F_q}(u) = b\},$$

$$J(r; a', b) = |\{x^r - a'x^{r-1} + \cdots + (-1)^rb \in F_q[x] \text{ is irreducible}\}|.$$

**Remark.** Let $\gamma$ be a primitive element of $F_{q^r}$ and assume that $a = N_{F_{q^r}/F_q}(\gamma^i)$, $b = N_{F_{q^r}/F_q}(\gamma^j)$. Then $|S_r(a, b)|$ is the cyclotomic number $(i, j)_{q-1}$ over $F_{q^r}$; see [22, p.247]. Cyclotomic numbers are closely related to Jacobi sums; see [2, §11.6].
Throughout this chapter, let $a = N_{F_q'/F_q}(\alpha)$, $b = N_{F_q'/F_q}(\beta)$. By (4.2) and (4.3),

$$\frac{N_3(\alpha, \beta)}{(q-1)^2} = |S_3(a, b)|. \quad (6.1)$$

Our method in this chapter consists of two steps: First we prove a peculiar connection between $S_3(a, b)$ and $T_3(b-a-1, ab)$ (Theorem 6.1). Then we use a result of Moisio [24] to express $T_3(b-a-1, ab)$ in terms of the number of rational points on a projective cubic.

**Theorem 6.1.** Let $a, b \in \mathbb{F}_q^*$. The mapping

$$\psi : S_3(a, b) \rightarrow T_3(b-a-1, ab)$$

$$u \mapsto u + u^{1+q}$$

is onto. More precisely, for each $v \in T_3(b-a-1, ab)$,

$$|\psi^{-1}(v)| = \begin{cases} q+1 & \text{if } a = 1 \text{ and } v = -1, \\ 1 & \text{otherwise.} \end{cases}$$

**Proof.** First we show that if $u \in S_3(a, b)$, then $u + u^{1+q}$ indeed belongs to $T_3(b-a-1, ab)$. Clearly,

$$N_{F_q'/F_q}(u + u^{1+q}) = N_{F_q'/F_q}(u(1+u)^q) = N_{F_q'/F_q}(u) N_{F_q'/F_q}(1+u) = ab.$$ We also have

$$b = N_{F_q'/F_q}(u + 1) = (u + 1)^{1+q+q^2}$$

$$= u^{1+q+q^2} + u^{1+q} + u^{q+q^2} + u^{q^2+1} + u^1 + u^q + u^{q^2} + 1$$

$$= N_{F_q'/F_q}(u) + 1 + \text{Tr}_{F_q'/F_q}(u + u^{1+q})$$

$$= a + 1 + \text{Tr}_{F_q'/F_q}(u + u^{1+q}).$$

So $\text{Tr}_{F_q'/F_q}(u + u^{1+q}) = b - a - 1$. 
Now let $\alpha \in \mathbb{F}_{q^3}^*$ such that $N_{\mathbb{F}_{q^3}/\mathbb{F}_q}(\alpha) = a$. Then $u \in \mathbb{F}_{q^3}^*$ satisfies $N_{\mathbb{F}_{q^3}/\mathbb{F}_q}(u) = a$ if and only if $u = \alpha x^{q-1}$ for some $x \in \mathbb{F}_{q^3}^*$.

For $v \in T_3(b - a - 1, ab)$, let $u \in \psi^{-1}(v)$ such that $N_{\mathbb{F}_{q^3}/\mathbb{F}_q}(u) = a$. Hence we can write $u = \alpha x^{q-1}$, for some $x \in \mathbb{F}_{q^3}^*$. We claim that $u = \alpha x^{q-1} \in \psi^{-1}(v)$ if and only if $x \in \mathbb{F}_{q^3}^*$ is a solution of

$$\alpha^{1+q}x^2 + \alpha x - vx = 0. \quad (6.2)$$

First assume $\alpha x^{q-1} \in \psi^{-1}(v)$. Then $\alpha x^{q-1} + (\alpha x^{q-1})^{1+q} = v$ and so $\alpha x^q + \alpha^{1+q}x^2 = vx$.

Next, we assume $x \in \mathbb{F}_{q^3}^*$ is a solution of (6.2). Then we have

$$\psi(\alpha x^{q-1}) = \alpha x^{q-1} + (\alpha x^{q-1})^{1+q} = v.$$

It remains to show that $u \in S_3(a, b)$. We only need to show that $N_{\mathbb{F}_{q^3}/\mathbb{F}_q}(u + 1) = b$.

We have

$$N_{\mathbb{F}_{q^3}/\mathbb{F}_q}(u + 1) = N_{\mathbb{F}_{q^3}/\mathbb{F}_q}(u) + Tr_{\mathbb{F}_{q^3}/\mathbb{F}_q}(u + u^{1+q}) + 1$$

$$= a + b - a - 1 + 1$$

$$= b.$$

This proves the claim.

The number of solutions $x \in \mathbb{F}_{q^3}$ of (6.2) is $q^{3-rank A}$, where

$$A = \begin{bmatrix}
    v & -\alpha & -\alpha^{1+q} \\
    -\alpha^{q+q^2} & \alpha^q & -\alpha^q \\
    -\alpha^{q^2} & -\alpha^{q^2+1} & \alpha^{q^2}
\end{bmatrix};$$
see [16, Proposition 2.1]. We have
\[
\det A = v^{1+q+q^2} - \alpha^{1+q+q^2} - \alpha^{2(1+q+q^2)} - \alpha^{1+q+q^2}(v^{1+q+q^2})
\]
\[
= N_{\mathbb{F}_q^3/\mathbb{F}_q}(v) - N_{\mathbb{F}_q^3/\mathbb{F}_q}(\alpha) - N_{\mathbb{F}_q^3/\mathbb{F}_q}(\alpha)^2 - N_{\mathbb{F}_q^3/\mathbb{F}_q}(\alpha) \text{Tr}_{\mathbb{F}_q^3/\mathbb{F}_q}(v)
\]
\[
= ab - a - a^2 - a(b - a - 1)
\]
\[
= 0.
\]
So \(\text{rank } A = 1 \) or 2. It is easy to see that
\[
\text{rank } A = \begin{cases} 
1 & \text{if } a = 1 \text{ and } v = -1, \\
2 & \text{otherwise.}
\end{cases}
\]
Hence the number of \( x \in \mathbb{F}_{q^3}^* \) of (6.2) is
\[
\begin{cases} 
q^2 - 1 & \text{if } a = 1 \text{ and } v = -1, \\
q - 1 & \text{otherwise.}
\end{cases}
\]
(6.3)

Now suppose \( x \in \mathbb{F}_{q^3}^* \) is a solution of (6.2). Then for \( \epsilon \in \mathbb{F}_q^* \), \( x\epsilon \) is also a solution since
\[
\alpha^{1+q}(x\epsilon)q^2 + \alpha(x\epsilon)^q - v(x\epsilon) = \epsilon(\alpha^{1+q}xq^2 + \alpha x^q - vx).
\]
Therefore, for \( v \in T_3(b - a - 1, ab) \), the number of \( u = \alpha xq^{-1} \in \psi^{-1}(v) \) is
\[
\begin{cases} 
q + 1 & \text{if } a = 1 \text{ and } v = -1, \\
1 & \text{otherwise.}
\end{cases}
\]

If \( a = 1 \) and \( v = -1 \in T_3(b - a - 1, ab) \), then \( \text{Tr}_{\mathbb{F}_q^3/\mathbb{F}_q}(-1) = b - 2 \) and \( N_{\mathbb{F}_q^3/\mathbb{F}_q}(-1) = b \), which imply \( b = -1 \). We have the following corollary.
Corollary 6.2. Let \( a, b \in \mathbb{F}_q^* \). Then

\[
[S_3(a, b)] = \begin{cases} 
|T_3(b - a - 1, ab)| + q & \text{if } a = 1 \text{ and } b = -1, \\
|T_3(b - a - 1, ab)| & \text{otherwise}.
\end{cases}
\tag{6.4}
\]

Combining (6.4) and (6.1), we arrive at a new formula for \( N_3(\alpha, \beta) \):

\[
\frac{N_3(\alpha, \beta)}{(q - 1)^2} = \begin{cases} 
|T_3(b - a - 1, ab)| + q & \text{if } a = 1 \text{ and } b = -1, \\
|T_3(b - a - 1, ab)| & \text{otherwise},
\end{cases}
\tag{6.5}
\]

where \( a = N_{\mathbb{F}_q^3/\mathbb{F}_q}(\alpha) \) and \( b = N_{\mathbb{F}_q^3/\mathbb{F}_q}(\beta) \).

Moisio [24] studied the number of irreducible polynomials over finite fields with prescribed trace and norm. The number \(|T_3(b - a - 1, ab)|\) in (6.5) is subject to further interpretations by the results of [24].

For \( c \in \mathbb{F}_q^* \), let \( \mathcal{B}_c \) be the affine cubic curve defined by

\[
\mathcal{B}_c : \quad y^2 + cy + xy = x^3
\]

and let \( \overline{\mathcal{B}}_c \) denote the projective closure of \( \mathcal{B}_c \). By Theorems 3.2 and 5.1 of [24], we have

\[
|T_3(b - a - 1, ab)| = \begin{cases} 
|\mathcal{B}_c(\mathbb{F}_q)|, \text{ where } c = \frac{ab}{(b - a - 1)^3}, & \text{if } b - a - 1 \neq 0, \\
q + 1 + \frac{1}{q} \sum_{x \in \mathbb{F}_q^3} e(\alpha\beta x^{3q-1}) & \text{if } b - a - 1 = 0,
\end{cases}
\tag{6.6}
\]

where \( e \) is the canonical additive character of \( \mathbb{F}_q^3 \).
We can also write

\[
|T_3(b - a - 1, ab)| = |T_3(b - a - 1, ab) \cap (\mathbb{F}_{q^3} \setminus \mathbb{F}_q)| + |T_3(b - a - 1, ab) \cap \mathbb{F}_q|
\]

\[
= 3J(3; b - a - 1, ab) + \left| \{v \in \mathbb{F}_q : 3v = b - a - 1, \ v^3 = ab \} \right|
\]

\[
= \begin{cases} 
3J(3; b - a - 1, ab) + 1 & \text{if } (b - a - 1)^3 = 27ab, \\
3J(3; b - a - 1, ab) & \text{otherwise.}
\end{cases}
\]  

(6.7)

If \((b - a - 1)^3 = 27ab\) and \(\text{char } \mathbb{F}_q \neq 3\), by Corollary 5.2 of [24],

\[
J(3; b - a - 1, ab) = \left\lfloor \frac{1}{3}(q + 1) \right\rfloor,
\]

so

\[
|T_3(b - a - 1, ab)| = 3 \left\lfloor \frac{1}{3}(q + 1) \right\rfloor + 1.
\]

This is also true when \(\text{char } \mathbb{F}_q = 3\) since in (6.6), we have \(\sum_{x \in \mathbb{F}_{q^3}} e(\alpha \beta x) = 0\) and \(3 \left\lfloor \frac{1}{3}(q + 1) \right\rfloor + 1 = q + 1\). Thus (6.7) can be made a little more explicit:

\[
|T_3(b - a - 1, ab)| = \begin{cases} 
3\left\lfloor \frac{1}{3}(q + 1) \right\rfloor + 1 & \text{if } (b - a - 1)^3 = 27ab, \\
3J(3; b - a - 1, ab) & \text{otherwise.}
\end{cases}
\]  

(6.8)

By (6.5) and (6.8) we obtain the following formula for \(N_3(\alpha, \beta)\):

\[
\frac{N_3(\alpha, \beta)}{(q - 1^2)} = \begin{cases} 
3\left\lfloor \frac{1}{3}(q + 1) \right\rfloor + q + 1 & \text{if } (a, b) = (1, -1), \\
3\left\lfloor \frac{1}{3}(q + 1) \right\rfloor + 1 & \text{if } (a, b) \neq (1, -1) \text{ and } (b - a - 1)^3 = 27ab, \\
3J(3; b - a - 1, ab) & \text{otherwise,}
\end{cases}
\]

where \(a = N_{\mathbb{F}_{q^3}/\mathbb{F}_q}(\alpha)\) and \(b = N_{\mathbb{F}_{q^3}/\mathbb{F}_q}(\beta)\).
In this chapter, we want to determine if $I(t; a, b)$ is positive with $a, b \in \mathbb{F}_q^*$ and integer $t > 0$. Namely, given $a, b$ and $t$, does there exist a monic irreducible polynomial $f \in \mathbb{F}_q[x]$ of degree $t$ such that $f(0) = a$ and $f(1) = b$? By (5.1), (5.8) and (5.9), we see that for $t = 1, 2$, we have $I(t; a, b) = 0$ or $1$ depending on certain conditions on $a$ and $b$. When $t = 3$, then by (5.13), $I(3; a, b) \geq 2$ since $|V(a, b)| \leq q - 2$. We will prove that $I(t; a, b) > 0$ for $t \geq 4$. Our proof is based on the relation between $I(t; a, b)$ and an estimate for $N_t(\alpha, \beta)$.

In some sense, the positivity of $I(t; a, b)$ ($t \geq 3$) is comparable with the Hansen-Mullen conjecture for irreducible polynomials [13] (proved by Wan [28] and Ham and Mullen [12]) which postulates that a prescribed degree and one prescribed coefficient can always be achieved by a monic irreducible polynomial in $\mathbb{F}_q[x]$ excluding two obvious non attainable cases.

We first look at $N_t(\alpha, \beta) = \left|\{(x, y) \in \mathbb{F}_q^* \times \mathbb{F}_q^* : x^{q-1} + \alpha y^{q-1} = \beta\}\right|$, where $\alpha, \beta \in \mathbb{F}_q^*$. The number of solutions of a diagonal equation can be expressed in terms of Gaussian sums and is given in (3.2). But note than in this expression, we considered all solutions in $\mathbb{F}_q^n$. For nonzero solutions, the computation is similar.

We now consider the solutions $(x, y) \in \mathbb{F}_q^* \times \mathbb{F}_q^*$ of the diagonal equation $x^{q-1} + \alpha y^{q-1} = \beta$. Let $\chi_1$ be the canonical additive character of $\mathbb{F}_{q^t}$ and let $\lambda$ be a multi-
plicative character of order \( q - 1 \) of \( \mathbb{F}_{q^t} \). Then

\[
N_t(\alpha, \beta) = \frac{1}{q^t} \sum_{x,y \in \mathbb{F}_{q^t}^*} \sum_{s \in \mathbb{F}_{q^t}^*} \chi_s(x^{q-1} + \alpha y^{q-1}) \overline{\chi_s}(\beta)
\]

\[
= \frac{(q^t - 1)^2}{q^t} + \frac{1}{q^t} \sum_{s \in \mathbb{F}_{q^t}^*} \chi_s(\beta) \sum_{x,y \in \mathbb{F}_{q^t}^*} \chi_s(x^{q-1}) \chi_s(\alpha y^{q-1})
\]

\[
= \frac{(q^t - 1)^2}{q^t} + \frac{1}{q^t} \sum_{j=0}^{q-2} \sum_{k=0}^{q-2} G(\lambda^j, \chi_1) G(\lambda^k, \chi_1) G(\lambda^{-j-k}, \overline{\chi_1})
\]

\[
= \frac{(q^t - 1)^2}{q^t} + \frac{1}{q^t} \sum_{j=0}^{q-2} \sum_{k=0}^{q-2} \lambda^j (-\beta) \lambda^k \left( -\frac{\beta}{\alpha} \right) G(\lambda^j, \chi_1) G(\lambda^k, \chi_1) G(\lambda^{-j-k}, \chi_1).
\]

(7.1)

Observe that by (2.6) and (2.7),

\[
|G(\lambda^j, \chi_1) G(\lambda^k, \chi_1) G(\lambda^{-j-k}, \chi_1)|
\]

\[
= \begin{cases} 
1 & \text{if } j, k, -j - k \text{ are all } \equiv 0 \pmod{(q-1)}, \\
q^t & \text{if exactly one of } j, k, -j - k \text{ is } \equiv 0 \pmod{(q-1)}, \\
q^{3t} & \text{if none of } j, k, -j - k \text{ is } \equiv 0 \pmod{(q-1)}. 
\end{cases}
\]

Thus

\[
N_t(\alpha, \beta) \geq \frac{(q^t - 1)^2}{q^t} - \frac{1}{q^t} \left[ 1 + 3(q-2)q^t + (q-2)(q-3)q^{2t} \right]
\]

\[
= q^t + 4 - 3q - (q^2 - 5q + 6)q^{2t}.
\]

And so we have the following lemma.

**Lemma 7.1.** Let \( \alpha, \beta \in \mathbb{F}_{q^t}^* \). Then we have

\[
N_t(\alpha, \beta) \geq q^t + 4 - 3q - (q^2 - 5q + 6)q^{2t}.
\]

**Remark.** Lemma 7.1 also follows from the Hasse-Weil bound ([27, Theorem V.2.3]). Since the genus of \( \mathcal{C} \) in (1.3) is \( \frac{1}{2}(q-2)(q-3) \) ([11, p.199]), the Hasse-Weil bound
gives $|C(F_q)| \geq q^t + 1 - (q - 2)(q - 3)q^{\frac{t}{2}}$. Thus by (1.4),

$$N_t(\alpha, \beta) \geq |C(F_q)| - 3(q - 1) \geq q^t + 4 - 3q - (q^2 - 5q + 6)q^{\frac{t}{2}}.$$  

We now prove the positivity of $I(t; a, b)$ for $t \geq 4$ in the next theorem.

**Theorem 7.2.** For $a, b \in F_q^*$ and $t \geq 4$ we have $I(t; a, b) > 0$.

**Proof.** If $q = 2$, then $a = b = 1$. Every irreducible polynomial $f \in F_2[x]$ with $\deg f > 1$ must have $f(0) = 1$ and $f(1) = 1$. Thus, $I(t; 1, 1) > 0$ for $t \geq 2$. Henceforth we assume $q \geq 3$.

Let $\alpha, \beta \in F_q^*$ such that $a = N_{F_q/F_q}(\alpha)$ and $b = N_{F_q/F_q}(\beta)$. By Theorem 4.1,

$$tI(t; a, b) = \frac{N_t(\alpha, \beta)}{(q-1)^2} - \sum_{r|t, r < t} r \sum_{a_1, b_1 \in F_q^*} I(r; a_1, b_1)$$

$$\geq \frac{1}{(q-1)^2} \left[ q^t + 4 - 3q - (q^2 - 5q + 6)q^{\frac{t}{2}} \right] - \sum_{r|t, r < t} r \left( \frac{t}{r} \right)^2 q^{r-2}$$

$$\geq \frac{1}{(q-1)^2} \left[ q^t + 4 - 3q - (q^2 - 5q + 6)q^{\frac{t}{2}} \right] - t^2 q^{-2} \sum_{r \leq \left\lfloor \frac{t}{2} \right\rfloor} q^r$$

$$= \frac{1}{(q-1)^2} \left[ q^t + 4 - 3q - (q^2 - 5q + 6)q^{\frac{t}{2}} \right] - t^2 q^{\left\lfloor \frac{t}{2} \right\rfloor} \frac{q^{\left\lfloor \frac{t}{2} \right\rfloor} + 1 - 1}{q-1}$$

$$\geq \frac{1}{(q-1)^2} \left[ q^t + 4 - 3q - (q^2 - 5q + 6)q^{\frac{t}{2}} \right] - \frac{t^2 q^{\lfloor \frac{t}{2} \rfloor}}{q(q-1)}$$

$$\geq \frac{1}{(q-1)^2} \left[ q^t + 4 - 3q - (q^2 - 5q + 6)q^{\frac{t}{2}} \right] - \frac{t^2 q^{\frac{t}{2}}}{(q-1)^2}$$

$$= \frac{1}{(q-1)^2} \left[ q^{\frac{t}{2}} (q^{\frac{t}{2}} - q^2 + 5q - 6 - t^2) + 4 - 3q \right].$$

Let $A(q, t) = q^{\frac{t}{2}} - q^2 + 5q - 6 - t^2$. Then

$$\begin{cases} \frac{\partial A}{\partial q} = \frac{t}{2} q^{\frac{t}{2}-1} - 2q + 5, \\
\frac{\partial A}{\partial t} = \frac{1}{2} q^{\frac{t}{2}} \ln q - 2t. \end{cases}$$
We have $A(5, 4) = 3$, $A(3, 8) = 17$ and

$$\frac{\partial A}{\partial q} > 0, \quad \frac{\partial A}{\partial t} > 0 \quad \text{for } q \geq 5, \ t \geq 4 \text{ or } q \geq 3, \ t \geq 8.$$ 

So when $q \geq 5$, $t \geq 4$ or $q \geq 3$, $t \geq 8$, we have $A(q, t) \geq 3$ and consequently

$$tI(t; a, b) \geq \frac{1}{(q - 1)^2}(q^2 \cdot 3 + 4 - 3q) > 0.$$ 

For $3 \leq q < 5$ and $4 \leq t < 8$, the positivity of $I(t; a, b)$ is checked directly using a computer.
A function \( f : \mathbb{F}_q \rightarrow \mathbb{F}_q \) is called **planar** if for every \( u \in \mathbb{F}_q^* \),

\[
x \mapsto f(x + u) - f(x)
\]

is a permutation of \( \mathbb{F}_q \). Planar functions were introduced by Dembowski and Ostrom [7] to describe certain affine planes. For further results on planar functions and related topics, see [6], [14], [18], [21]. Recently, planar functions have found important applications in cryptography where they are called **perfect nonlinear functions**; see [26]. Constructions of perfect nonlinear functions and their close relatives **almost perfect nonlinear functions** have been attracting much attention for the past decade, see [3], [8], [9], [10], [15], [19].

Observe that planar functions exist only when \( q \) is odd.

**Lemma 8.1.** Let \( p \) be an odd prime and \( n \) be a positive integer. Let

\[
f(x) = x^{p^m+1} + \beta x^2 \in \mathbb{F}_{p^n}[x],
\]

where \( m > 0 \) and \( \beta \in \mathbb{F}_{p^n}^* \). Let \( t = \frac{n}{(m,n)} \) and \( q = p^{(m,n)} = p^\frac{n}{t} \) (so \( q^t = p^n \)). Then \( f \) is a planar function on \( \mathbb{F}_{q^t} \) if and only if \( N_t(1, -2\beta) = 0 \), i.e., if and only if \( x^{q^{-1}} + y^{q^{-1}} = -2\beta \) has no solution \( (x, y) \in \mathbb{F}_{q^t}^* \times \mathbb{F}_{q^t}^* \).

**Proof.** Let \( u \in \mathbb{F}_{q^t} \). Since \( f(u) \) is constant, then \( f(x + u) - f(x) \) is a permutation of
If and only if $f(x+u) - f(x) - f(u)$ is a permutation of $\mathbb{F}_{q^t}$. We have

$$f(x+u) - f(x) - f(u) = ux^{p^m} + u^{p^m} x + 2\beta ux$$

$$= ux(x^{p^m-1} + u^{p^m-1} + 2\beta).$$

Now $f(x+u) - f(x) - f(u)$ is a $p$-polynomial. By [22, Theorem 7.9], it is a permutation of $\mathbb{F}_{q^t}$ if and only if $x = 0$ is its only root, i.e., if and only if

$$x^{p^m-1} + u^{p^m-1} \neq -2\beta \quad \text{for all } x, u \in \mathbb{F}_{q^t}^*.$$

But $\mathbb{F}_{q^t}^{*}(p^m-1) = \mathbb{F}_{q^t}^{*}(p^m-1,q^t-1) = \mathbb{F}_{q^t}^{*}(p^m,n-1)$. And so $f(x+u) - f(x) - f(u)$ is a permutation of $\mathbb{F}_{q^t}$ if and only if

$$x^{q-1} + u^{q-1} \neq -2\beta \quad \text{for all } x, u \in \mathbb{F}_{q^t}^*.$$

Therefore, $f$ is a planar function on $\mathbb{F}_{q^t}$ if and only if $N_t(1,-2\beta) = 0$.

In Lemma 8.1, if $t = 1$, then $f(x) = (\beta + 1)x^2$ on $\mathbb{F}_q$, which is not interesting; if $t \geq 3$, we know from chapter 7 that $N_t(1,-2\beta) > 0$, so Lemma 8.1 does not produce any planar function. The only interesting case in this Lemma is when $t = 2$. Let $t = 2$ and let $b = N_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\beta)$. Then $N_{\mathbb{F}_{q^2}/\mathbb{F}_q}(-2\beta) = 4b$. By (5.11) we have

$$\frac{N_2(1,-2\beta)}{(q-1)^2} = 1 - \eta((4b - 2)^2 - 4)$$

$$= 1 - \eta(b(b-1)).$$

Combining the above equation and Lemma 8.1, we have the following proposition.

**Proposition 8.2.** Let $p$ be an odd prime and let $n,m$ be positive integers such that $(m,n) = \frac{n}{2}$. Put $q = p^\frac{n}{2}$ (so $p^n = q^2$). Let $f(x) = x^{p^m+1} + \beta x^2 \in \mathbb{F}_{q^2}[x]$, where $\beta \in \mathbb{F}_{q^2}^*$. Then $f$ is a planar function on $\mathbb{F}_{q^2}$ if and only if $\eta(b(b-1)) = 1$, where $b = N_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\beta)$. 

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Suppose $f$ satisfy the assumptions in Proposition 8.2. We know that $f$ is a planar function on $\mathbb{F}_{q^2}$ if and only if $N_2(1, -2\beta) = 0$. We count the number of $\beta$ so that $f$ is planar. Write $H = \mathbb{F}_{q^2}^{x(q-1)}$. Then

$$\{\beta' \in \mathbb{F}_{q^2}^* : N_2(1, \beta') > 0\} = \mathbb{F}_{q^2}^* \cap (H + H) = H(1 + H\{−1\}).$$

Let $q$ be odd and $x, y \in H\{−1\}$. We have

$$\frac{1 + y}{1 + x} \in H \iff \left(\frac{1 + y}{1 + x}\right)^{q+1} = 1$$
$$\iff (1 + y)^{q+1} = (1 + x)^{q+1}$$
$$\iff (1 + y)(1 + y^q) = (1 + x)(1 + x^q)$$
$$\iff 2 + y + y^q = 2 + x + x^q$$
$$\iff y + y^{-1} = x + x^{-1}$$
$$\iff (y - x)(xy - 1) = 0$$
$$\iff y = x \text{ or } y = x^{-1}.$$ 

Therefore, if $x \neq 1$, then there are precisely two $y \in H$ ($y = x$ or $x^{-1}$) such that $H(1 + x) = H(1 + y)$. If $x = 1$, then there is exactly one $y \in H$ ($y = x$) such that $H(1 + x) = H(1 + y)$. Thus,

$$|H(1 + H\{−1\})| = |H| \left(\frac{1}{2}(|H| - 2) + 1\right) = \frac{1}{2}|H|^2 = \frac{1}{2}(q + 1)^2.$$ 

Hence the number of $\beta$ in Proposition 8.2 so that $f$ is a planar function is

$$q^2 - 1 - \frac{1}{2}(q + 1)^2 = \frac{1}{2}(q + 1)(q - 3).$$
REFERENCES


